Solution to Problem # 675

Problem: Let a, b, c be distinct integers. Is there a polynomial P with integer coefficients such that the relations

$$P(a) = b, \quad P(b) = c, \quad \text{and } P(c) = a$$

hold?

Solution. We will show that there is no such polynomial. If such a polynomial P did exist, let the polynomial $Q(x, y)$ of two variables be defined by

$$Q(x, y) = \frac{P(x) - P(y)}{x - y},$$

so that it is clear that Q has integer coefficients. Then by hypothesis

$$Q(a, b) = \frac{b - c}{a - b}, \quad Q(b, c) = \frac{c - a}{b - c}, \quad Q(c, a) = \frac{a - b}{c - a},$$

so that we have

$$Q(a, b) \cdot Q(b, c) \cdot Q(c, a) = 1.$$

Since each of the three factors on the left is an integer, it follows that they are each equal t ± 1. Consequently

$$(a - b) = \pm(b - c) = \pm(c - a).$$

If the first ambiguous sign is negative, then $a - b = -b + c$, i.e. $a = c$ which contradicts the assumption that a, b, c are distinct. Similarly, if the second ambiguous sign is negative, we have $b = c$. Therefore we have

$$(a - b) = (b - c) = (c - a).$$

But then the sum of the three equal numbers in the above equation is zero, which means again that $a = b = c$. $\square$

This proof can be generalized to show that if $a_1, \ldots, a_n$ are distinct integers, there is no polynomial P with integer coefficients such that $P(a_k) = a_{k+1}$ for $1 \leq k \leq n - 1$ and $P(a_n) = P(a_n)$. This problem appeared in the USAMO of 1974.