Solution to Problem # 672

Problem:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, \cdots$$

is the celebrated *Fibonacci sequence* in which each term beginning with the third is equal to the sum of th two previous terms. As you can see the 15th term of the sequence ends with a 0. Prove that there is a term in the Fibonacci sequence that ends with 0000.

Solution. More generally, one can show that for a positive integer N, there are infinitely many terms of the Fibonacci sequence that are divisible by N. The problem as stated corresponds to $N = 10000$. Consider the sequence $\{x_k\}_{k=1}^{\infty}$ where x_k is the remainder on dividing the k-th term of the Fibonacci sequence by N, so that $0 \leq x_k \leq N-1$. Then the pairs (x_n, x_{n+1}) can assume N^2 different values, but there are infinitely many values of n, so there are two positive integers m, n with $m > n$ such that $x_m = x_n$ and $x_{m+1} = x_{n+1}$. By the recurrence relation for the Fibonacci sequence, $x_{m-1} = x_{n-1}, x_{m-2} = x_{n-2}, \ldots, x_2 = x_{2+n-m}, x_1 = x_{1+n-m}$. Therefore $x_{2+n-m} = x_{1+n-m} = 1$, and therefore $x_{n-m} = 0$, which shows that there is a term of the Fibonacci sequence divisible by N.

The same argument, appropriately modified shows that for each positive integer k, we have $x_{k(n-m)} = 0$, i.e. there are infinitely many terms of the Fibonacci sequence divisible by N.

Kolbi Esch showed by computation that the 7500th term of the Fibonacci sequence is the first one to end in 4 zeros. This number is : 1142396523 1520587047 2204889286 5690419848 7186633317 5607979590 3059573826 3643588305 2639643210 8051699142 9937628886 2295553401 4664444274 4473185460 7783029347 4380700224 8109695741 2087824111 5918999465 1520930091 2020351012 6935052360 9417276542 2096822611 6815054479 0025062794 2090915037 0208857433 8650460569 2955924986 6644323980 7989522593 0725621586 4094746865 6887645879 3562013015 9484187249 1497556389 5558172775 0834905833 0498007583 8142701233 2972435323 3156029127 9109683700 5273481119 2660492733 3753944726 9219158448 9489590970 2544409142 2277838243 9339334175 6246602915 8877845625 0479185237 8983091123 1882998435 8216337347 5490143365 1748649664 3224502773 3800420711 7436059719 2343056318 4892870384 4700473092 2073980870 0729907060 6750862403 8407888471 2940489122 9415349139 8930715643 6401701728 3737912796 9101176561 4505869457 1546027678 0809807889 6642728183 1686571172 4985646554 5593053343 4031899461 2185260719 0420089603 1126900012 2672589731 2834196080 9830336726 0382379660 4022618865 7495221178 3683104453 3342816844 2599444730 6306414660 0325190550 7950431356 2694958935 7541187961 5763297897 0220780288 1689921816 9970892297 1417067735 1449294611 9363908144 5200786881 5493311503 8121607370 5417531166 7866346904 6920641861 1524663013 8541980452 8480672073 5273715046 8887049168 2185527754 3026346215 3552863958 5426316825 1068150374 9888516205 0119694390 5031285049 0776284438 0405213450 7022504682 4832933962 1526818662 0124762379 7446680921 6603531455 3541731537 2459462564 2286185257 3006230492 3222596303 4229435082 7184840607 5099692893 2832036009 3204783447 8609558063 9635072334 1261564285 6494530079 4908915416 5288839814 4426773393 4479469188 1510389855 7655827167 74490000 □