Solution to Problem # 671

Problem:

(a) Let A_1, A_2, A_3, A_4, A_5 be five points in the plane. Show that

$$2(A_1A_2+A_2A_3+A_3A_4+A_4A_5+A_5A_1) \geq A_1A_3+A_2A_4+A_3A_5+A_4A_1+A_5A_2.$$

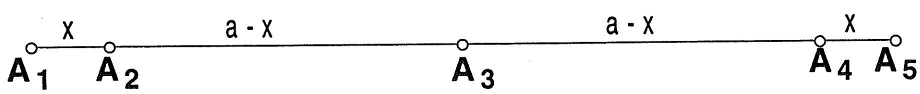
(b) Show that the coefficient 2 in this inequality is the best possible, i.e., there are 5 points A_1, A_2, A_3, A_4, A_5 in th plane such that

$$1.9(A_1A_2+A_2A_3+A_3A_4+A_4A_5+A_5A_1) < A_1A_3+A_2A_4+A_3A_5+A_4A_1+A_5A_2.$$

Solution. By the triangle inequality, for any three points X, Y, Z in the plane $XY + YZ \geq XZ$. Therefore

$$A_1A_2 + A_2A_3 \geq A_1A_3, \quad A_2A_3 + A_3A_4 \geq A_2A_4 \qquad A_3A_4 + A_4A_5 \geq A_3A_5$$
$$A_4A_5 + A_5A_1 \geq A_4A_1, \quad A_5A_1 + A_1A_2 \geq A_5A_2.$$

Adding these 5 inequalities, (a) follows. For (b), choose points A_1, A_2, A_3, A_4, A_5 to be collinear and appear in that order on a line (see figure below). Let $A_1A_3 = A_3A_5 = a$ and $A_1A_2 = A_4A_5 = x$. Then $A_2A_3 = A_3A_4 = a - x$.



Then

$$A_1A_2 + A_2A_3 + A_3A_4 + A_4A_5 + A_5A_1 = 4a \qquad \text{and}$$
$$A_1A_3 + A_2A_4 + A_3A_5 + A_4A_1 + A_5A_2 = 8a - 4x.$$

For any $p < 2$, the inequality $p \cdot 4a < 8a - 4x$ us equivalent to $x < (2-p)a$. If we choose an x satisfying the latter inequality, for each $p < 2$ there are 5 points A_1, A_2, A_3, A_4, A_5 in the plane such that

$$p(A_1A_2+A_2A_3+A_3A_4+A_4A_5+A_5A_1) < A_1A_3+A_2A_4+A_3A_5+A_4A_1+A_5A_2.$$

In particular, this is true for $p = 1.9$.

□