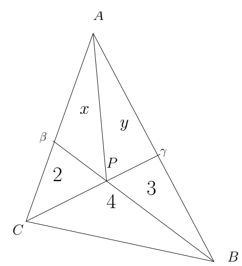
Solution to Problem # 670

Problem:

Two straight lines are drawn, each from a vertex of a triangle to the opposite side. This divides the triangle into four pieces: three smaller triangles and one quadrilateral. The areas of the smaller triangle are shown in the figure above. Find the area of the quadrilateral.

Solution. Join the point of intersection P of the two given lines with the remaining vertex A. This divides the quadrilateral $A\beta P\gamma$ into two triangles $\Delta A\beta P$ and $\Delta AP\gamma$, and let their areas be x and y respectively:



The two triangles $\Delta A\beta P$ and βPC of areas x and 2 respectively share a common altitude, namely the perpendicular from P to the side AC. Therefore, using the fact that the area of a triangle if $\frac{1}{2} \times \text{base} \times \text{height}$, we have $\dfrac{x}{2} = \dfrac{\text{area}\,\Delta AP\beta}{\text{area}\,\Delta\beta BC} = \dfrac{A\beta}{\beta C}$. The same reasoning applied to $\Delta A\beta B$ and $\Delta\beta BC$ gives us $\dfrac{x+y+3}{2+4} = \dfrac{A\beta}{\beta C}$ so that we have $\dfrac{x}{2} = \dfrac{x+y+3}{6}$, i.e., $2x - y = 3$ Using the same argument with the triangles $\Delta AP\gamma, \Delta\gamma PB$ and $\Delta AC\gamma, \Delta\gamma CB$, we hav $\dfrac{y}{3} = \dfrac{A\gamma}{\gamma B} = \dfrac{x+y+2}{3+4}$. So we get $3x - 4y = -6$. Solving $2x - y = 3$ and $3x - 4y = -6$ simultaneously, we find $x = \dfrac{18}{5}$ and $y = \dfrac{21}{5}$. Therefore the area of the quadrilateral is $x + y = \dfrac{39}{5}$. □