Solution to Problem # 669

Problem:

Consider 999 consecutive perfect cubes starting with 2^3:

$$2^3 = 8, 3^3 = 27, 4^3 = 64, 5^3 = 125, \cdots 1000^3 = 1000000000.$$

Form a fraction, whose numerator is the product of the numbers which are one less than these cubes, i.e.,

$$7 \cdot 26 \cdot 63 \cdot 124 \cdot \dots \cdot 999999999,$$

and the denominator is the product of the numbers which are one more than these cubes, i.e.,

$$9 \cdot 28 \cdot 65 \cdot 126 \cdot \dots \cdot 1000000001.$$

Reduce this fraction to its lowest terms.

Solution. Since $x^3 - 1 = (x-1)(x^2+x+1)$ and $x^3+1 = (x+1)(x^2 - x + 1)$, the numerator can be factored as

$$\begin{aligned}&(2^3-1)(3^3-1)\dots(1000^3-1)\\ &= (2-1)(2^2+2+1)(3-1)(3^2+3+1)\dots(1000-1)(1000^2+1000+1)\\ &= 1\cdot 2\cdot \dots \cdot 999 \cdot (2^2+2+1)\cdot(3^2+3+1)\cdot \dots \cdot (1000^2+1000+1).\end{aligned}$$

and similarly the denominator can be factored as

$$\begin{aligned}&(2^3+1)(3^3+1)\dots(1000^3+1)\\ &= (2+1)(2^2-2+1)(3+1)(3^2-3+1)\dots(1000+1)(1000^2-1000+1)\\ &= 3\cdot 4\cdot \dots \cdot 1001 \cdot (2^2-2+1)\cdot(3^2-3+1)\cdot \dots \cdot (1000^2-1000+1).\end{aligned}$$

Since $x^2+x+1 = (x+1)^2 - (x+1) + 1$, we have $2^2+2+1 = 3^2-3+1, 3^2+3+1 = 4^2-4+1$ etc. Therefore, in the fraction we can cancel out each factor of the numerator and of the denominator, except $2\cdot(1000^2+1000+1)$ in the numerator and $(2^2-2+1)\cdot 1000 \cdot 10001$. Removing the common factor 6 from both, we have the value $\frac{333667}{500500}$ which is easily verified to be in the lowest terms (e.g. by factoring or Euclidean algorithm.) □