Solution to Problem # 668

Problem:

Find all equilateral triangles (i.e. triangles whose all three sides are of equal length) in the xy-plane with the following property: for each of the three vertices, both the x and y-coordinates are integers.

***Solution*.** Let A, B, C denote the vertices of the triangle, and suppose the coordinates of these vertices are $A = (x_1, y_1)$, $B = (x_2, y_2)$ and $C = (x_3, y_3)$. By hypothesis $AB = BC = CA = s$. Then $s^2 = AB^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$. We compute the area of the triangle ABC in two different ways:

(1) The height of the equilateral triangle ABC is $AB \cdot \sin 60^\circ = \frac{\sqrt{3}}{2}s$, so the area of ABC is

$$\frac{1}{2} \times \text{base} \times \text{height} = \frac{\sqrt{3}}{4}s^2 = \frac{\sqrt{3}}{4}\left((x_2 - x_1)^2 + (y_2 - y_1)^2\right).$$

Since the vertices have integer coordinates, it follows that the area of the triangle ABC is an *irrational number*.

(2) Recall that the area of the triangle is also given by $\frac{1}{2}\left|\vec{AB} \times \vec{AC}\right|$, where $\times$ is the cross product of vectors. Now

$$\vec{AB} = (x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j}, \quad \vec{AC} = (x_3 - x_1)\mathbf{i} + (y_3 - y_1)\mathbf{j}$$

so

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_2 - x_1 & y_2 - y_1 & 0 \\ x_3 - x_1 & y_3 - y_1 & 0 \end{vmatrix} = \left((x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1)\right)\mathbf{k},$$

so the area is given by $\frac{1}{2}\left|(x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1)\right|$, which is a *rational number*, since the coordinates $x_1, y_1, x_2, y_2, x_3, y_3$ are integers.

This contradiction shows that **there is no such triangle**. $\square$