Solution to Problem # 667

Problem: Let a, b, c be nonzero real numbers such that

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{1}{a+b+c}.$$

Show that

$$\frac{1}{a^{2025}}+\frac{1}{b^{2025}}+\frac{1}{c^{2025}}=\frac{1}{a^{2025}+b^{2025}+c^{2025}}.$$

Solution. Reducing the given condition on a, b, c to a common denominator and multiplying we have

$$(a+b+c)(ab+bc+ca)=abc.$$

Multiplying and moving everything to the left we get

$$a^2b+2abc+a^2c+b^2a+b,+bc^2+c^2a=0.$$

Writing in descending powers of a we have

$$a^2(b+c)+a(b^2c+2bc+c^2)+(b^2c+cb^2)=0.$$

Factoring the left hand side we have

$$(b+c)\left[a^2+a(b+c)+bc\right]=0.$$

That is

$$(a+b)(b+c)(c+a)=0.$$

One of the factors in the above product must vanish. Renaming the numbers, we may assume without loss of generality that $a+b=0$, so that we have $b=-a$ and therefore $b^{2025}=-a^{2025}$. Consequently

$$\frac{1}{a^{2025}}+\frac{1}{b^{2025}}+\frac{1}{c^{2025}}=\frac{1}{a^{2025}}-\frac{1}{a^{2025}}+\frac{1}{c^{2025}}=\frac{1}{c^{2025}}$$

and

$$\frac{1}{a^{2025}+b^{2025}+c^{2025}}=\frac{1}{a^{2025}-a^{2025}+c^{2025}}=\frac{1}{c^{2025}}.$$

$\square$