Solution to Problem # 666

Problem:

Find five positive whole numbers a, b, c, d, e such that there is no subset with a sum divisible by 5.

***Solution*.** We claim that no such collection of positive whole numbers exists. If such numbers existed, consider the sums $a, a+b, a+b+c, a+b+c+d$ and $a+b+c+d+e$. The remainders that these sums leave on division by 5 must have one of the values $1, 2, 3$ or 4, since we have assumed that there is no subset whose sum is divisible by 5. But then there are two sums which leave the same remainder, since there are 4 possible remainders and 5 sums. For definiteness, let us say that the second sum $a+b$ and the last sum $a+b+c+d+e$ leave the same remainder r if divided by 5. Calling the respective quotients m and n then we have

$$a+b = 5m+r, \quad a+b+c+d+e = 5n+r.$$

Then

$$(c+d+e) = (a+b+c+d+e) - (a+b) = 5(n-m),$$

and this shows that there is a subset of three elements whose sum is divisible by 5. Clearly this same argument applies whichever be the two elements in the collection $a, a+b, a+b+c, a+b+c+d, a+b+c+d+e$. with the same remainder on division by 5. $\square$